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On the interaction of surface and internal gravity waves: uniformly valid solution by extended stationary phase

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Using simple Fourier transform techniques and extensions to stationary-phase methods, the behaviour of surface gravity waves is determined near triply coalescing roots of the dispersion relation. It is shown that the amplitude of the surface wave is proportional to $(\partial U/\partial x)^{-\frac{1}{4}}$ at the location of the triple root. Far from the triple root it satisfies conservation of action. The internal wave is modelled simply by its surface current U.

Asymptotic orders of magnitude are also given for the case $\partial U/\partial x = 0$ at the triple root.

1. Introduction

In an earlier paper (Gargett & Hughes 1972) it was shown that the constant-action solution for interacting surface and internal gravity waves displays singularities and that these occur at coalescing roots of the surface-wave frequency dispersion relation. The present paper discusses the removal of these singularities by using simple extensions to ordinary stationary-phase techniques.

It has been shown by many researchers (Shand 1953; Lafond 1962; Perry & Schimke 1965; Apel *et al.* 1975) that striking patterns are produced in a surface wind-wave field by the presence of internal waves. The interaction can be strong enough to produce rough, breaking areas alternating with very smooth 'slicky' regions in winds up to 20 knots, or particular enough (apparently) to enhance only a narrow part of the wind-wave spectrum and produce long-crested semi-regular waves at particular parts of the internal wave field (Gargett & Hughes 1972). The present problem emerged from an attempt to describe the interaction theoretically, and we contend that within the basic assumption of the theory the proper representation of any surface wave field suffering this kind of perturbation is in terms of the results given here.

In our model, the surface wave field is assumed to be irrotational, the potential is expressed as a Fourier integral and the equations of motion are linearized. Our solution represents the 'free' wave field, that is, each component separately satisfies the equations of motion and remains bounded as $|x|$, $|y|$ or $|t| \to \infty$. Other boundary and initial conditions can be obtained by simple Fourier superposition. To obtain our solution we have merely performed one of the integrations in the triple Fourier integral. In the absence of a perturbing current this process would lead to the usual frequency dispersion relation and the condition that the most general three-dimensional (x, y and t) transform contains a one-dimensional delta function of arbitrary constant 'area'. In the present case we obtain the well-known Doppler-shifted approximate dispersion relation and an approximate delta function whose 'area' is non-constant (but exhibits constant action flux) and in some cases non-arbitrary owing to the presence of caustics.

Because of the linearization, finite amplitudes are obtained by superposition, and randomness and broad-band effects can be straightforwardly incorporated. The work by Holliday (1973), on the other hand, indicates that finite amplitude self-interactions (or amplitude dispersion effects) also remove the singularity. Unfortunately it is not clear at present how general this result is since his finite amplitude model is not readily extendible to include broad-band, random effects. In view of this difficulty, we consider the present approach worthy of attention not only in its application to the present problem but also because it may be easily extended to include other effects (such as surface tension, simple wind growth terms, depth dependence, etc.). It also provides a basis on which to compare numerical treatments which include nonlinear surface wave–wave interactions (West, Thomson & Watson 1975).

The basic assumption explicitly ignores all cross-spectral coupling of surface-wave components, in particular, the important resonant couplings. It is known that the time scale for these interactions is $\sim T/\overline{(\nabla\zeta)^2}$, where T is a characteristic period of the surface waves and $\overline{(\nabla\zeta)^2}$ is the mean-square slope of the interacting components (Phillips 1966, § 3.8). This can be made arbitrarily large by restricting the surface-wave amplitudes (more precisely, slopes) to sufficiently small values. On the other hand, the internal-wave interaction theory to be presented is linear in the surface-wave amplitude and it can be shown that its characteristic interaction time scale is $\sim \lambda_I T/\lambda_s$, where λ_I/λ_s is the ratio of internal-wave and surface-wave wavelengths. This is independent of surface-wave amplitude and thus will dominate the cross-spectral coupling terms for small enough surface slopes and short enough times.

At the centre of the analysis is the technique of evaluating integrals with many nearby stationary-phase points. This technique has undergone steady research in the past two decades, with the work by Bleistein (1967) and Ursell (1972) being most relevant to the present paper. (See also Ludwig & Olver 1970.)

In very recent years increasing use has been made of Pearcey's cusp functions, particularly by acousticians calculating approximate solutions to the wave equation in a varying medium. These functions are not well known and since they form the basis of the surface-wave solution obtained here, an appendix has been included in which we have defined some of their properties.

The internal wave is modelled by its most basic simple form: a non-dispersive field with a horizontal current that is uniform over all depths significantly affected by the surface waves. This is an idealization of a two-layer wave system in which the depth of the upper layer is large compared with the surface-wave wavelength and small compared with the internal-wave wavelength.

The analysis allows the internal-wave current to be arbitrarily large (but less than the internal-wave phase velocity) and it is asymptotically correct as the ratio of the internal-wave wavelength and the surface-wave wavelength becomes infinite. The present technique provides an alternative approach to a similar problem studied by Smith (1976) and it also provides an extension to his solution (for the case $\partial U/\partial x = 0$ at the triple root).

2. Approximation of the equations of motion

In the interests of simplification of this somewhat complicated problem, we shall ignore surface tension and wind growth/decay effects, and we shall assume that the depth is infinite. Let us also assume that potential flow exists. Then we may use the standard equations

$$\nabla^2\phi = 0,$$

$$\left.\begin{aligned} &\partial\phi/\partial t + \tfrac{1}{2}q^2 + gz = 0 \quad &(1)\\ &\frac{\partial\zeta}{\partial t}+\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y}-\frac{\partial\phi}{\partial z} = 0 \quad &(2)\end{aligned}\right\} \quad \text{at} \quad z=\zeta,$$

$$\phi \to 0 \quad \text{as} \quad z \to -\infty,$$

where ϕ is the velocity potential, ζ is the water height above an arbitrary reference level, $\mathbf{q}$ is the velocity vector and g is the acceleration due to gravity. Let us also assume that the surface wave does not affect the internal wave appreciably (or that the time scale of that variation is much longer than any other of interest). Then, with

$$\phi = \phi_s + \phi_I, \quad \zeta = \zeta_s + \zeta_I,$$

we have

$$\left.\begin{aligned} &\partial\phi_I/\partial t + \tfrac{1}{2}q_I^2 + g\zeta_I = 0 \quad &(3a)\\ &\frac{\partial\zeta_I}{\partial t}+\frac{\partial\phi_I}{\partial x}\frac{\partial\zeta_I}{\partial x}+\frac{\partial\phi_I}{\partial y}\frac{\partial\zeta_I}{\partial y}-\frac{\partial\phi_I}{\partial z} = 0 \quad &(3b)\end{aligned}\right\} \quad \text{at} \quad z=\zeta_1.$$

Expand (1) and (2) about $z = \zeta_1$ and ignore terms of order $\zeta_s^n\,\phi_s^m$ where $n+m \geqslant 2$. With no loss of generality for this problem let the internal wave propagate along the x axis, i.e.

$$\partial\phi_I/\partial y = \partial\zeta_I/\partial y = 0.$$

Then with W the vertical internal-wave current and U the horizontal current,

$$\nabla^2\phi_s = 0,$$

$$\left\{g+\left[\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial x}+W\frac{\partial W}{\partial z}\right]_{\zeta_I}\right\}\zeta_s+\left[\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}+W\frac{\partial}{\partial z}\right)\phi_s\right]_{\zeta_I} = 0,$$

$$\left\{\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right\}_{\zeta_I}\zeta_s+\left(\frac{\partial W}{\partial x}\frac{\partial\zeta_I}{\partial x}+\frac{\partial U}{\partial x}\right)_{\zeta_I}\zeta_s-\left(\frac{\partial\phi_s}{\partial z}\right)_{\zeta_I}+\left(\frac{\partial\zeta_I}{\partial x}\frac{\partial\phi_s}{\partial x}\right)_{\zeta_I} = 0.$$

The subscript ζ_I indicates that quantities dependent on z are to be evaluated at $z = \zeta_I(x,t)$. It should be noted that we have retained the full nonlinear expressions for the internal-wave currents, which allows us to apply our results to cases where the currents are not vanishingly weak. (We must, however, avoid regions where vorticity is important since we have used a velocity potential to describe the internal wave. For two-layer systems this is possible since the vorticity resides on the interface between the two fluids and is non-existent at the free surface.) The first of these three equations is solved exactly if ϕ_s is defined by an integral transform:

$$\phi_s = \iiint_{-\infty}^{+\infty} \Phi(k_1, k_2, \sigma) \exp\left[ik_1 x + ik_2 y + i\sigma t + z(k_1^2 + k_2^2)^{\frac{1}{2}}\right] dk_1\, dk_2\, d\sigma.$$

This expression can be inserted in the second equation, the operation

$$\frac{\partial}{\partial t} + U(x, \zeta_I(x,t), t)\frac{\partial}{\partial x} + W(x, \zeta_I(x,t), t)\frac{\partial}{\partial z}$$

can be performed under the integral sign, and z can then be put equal to $\zeta_I(x,t)$ in the exponent in ϕ_s. This allows us to solve for ζ_s by simple division of the term

$$g + \left[\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + W\frac{\partial W}{\partial z}\right]_{\zeta_I}.$$

Using DW/Dt to represent the substantial derivative in the square brackets in this term, k for $(k_1^2 + k_2^2)^{\frac{1}{2}}$ and the subscript 0 to mean evaluation at $z = \zeta_I$, we obtain

$$\zeta_s = -\iiint_{-\infty}^{+\infty} \left[\frac{i\sigma + iU_0 k_1 + W_0 k}{g + DW/Dt}\right] \times \Phi(k_1, k_2, \sigma) \exp\{i(k_1 x + k_2 y + \sigma t) + k\zeta_I(x,t)\}\, dk_1\, dk_2\, d\sigma.$$

Since ζ_I and the term in the square brackets are functions of x and t, this is *not* a simple Fourier transform definition for ζ_s. We may, however, substitute this expression along with the expression for ϕ_s into the third equation and obtain the following integral equation for Φ:

$$\iiint_{-\infty}^{+\infty} \Phi(k_1, k_2, \sigma) \exp\{i(k_1 x + k_2 y + \sigma t) + \zeta_I k\} \times \left[\frac{i\sigma + iU_0 k_1 + W_0 k}{g + DW/Dt}\left\{i\sigma + \frac{\partial \zeta_I}{\partial t}k + ik_1 U_0 + kU_0\frac{\partial \zeta_I}{\partial x} + \frac{\partial W_0}{\partial x}\frac{\partial \zeta_I}{\partial x} + \left(\frac{\partial U}{\partial x}\right)_0\right\} + \left(\frac{\partial}{\partial t} + U_0\frac{\partial}{\partial x}\right)\left(\frac{i\sigma + iU_0 k_1 + W_0 k}{g + DW/Dt}\right) + k - ik_1\frac{\partial \zeta_I}{\partial x}\right] dk_1\, dk_2\, d\sigma = 0.$$

Again we have a functional dependence on x and t in ζ_I and in the terms in the square brackets. It is this dependence which requires us to use the approximate stationary-phase techniques to determine Φ.

We shall now reduce the complexity of the expressions in the square brackets by ignoring the very small terms. Let us assume that the surface current U_0 and the propagation velocities of the internal and surface waves are of the same order. Let us designate the internal-wave wavenumber by k_I and let us define $\mu = k/k_I$. We shall assume $\mu \gg 1$ and retain terms which are $O(1)$ and $O(\mu^{-1})$ only. From (3a), $k_I\zeta_I = O(\mu^{-1})$, and from (3b), $W_0/U_0 = O(\mu^{-1})$. This leads to

$$DW/Dt = O(g\mu^{-2}),$$

and so DW/Dt can be ignored compared with g. Similar comparisons and some algebraic rearranging result in

$$\zeta_s = -\iiint_{-\infty}^{+\infty}\left[\frac{i\sigma+iU_0k_1+W_0k}{g}\right]\Phi\exp\{i(k_1x+k_2y+\sigma t)+\zeta_I k\}\,dk_1\,dk_2\,d\sigma, \quad (4a)$$

$$\iiint_{-\infty}^{+\infty}\Phi\exp\{i(k_1x+k_2y+\sigma t)+\zeta_I k\}\Big\{gk-(\sigma+k_1U_0)^2 + \left[i(\sigma+k_1U_0)\left(\frac{\partial U_0}{\partial x}+2W_0k\right)+2ik_1\frac{\partial U_0}{\partial t}+2ik_1U_0\frac{\partial U_0}{\partial x}\right]\Big\}\,dk_1\,dk_2\,d\sigma = 0. \quad (4b)$$

The expression in the square brackets in (4b) is $O(\mu^{-1})$ compared with gk. Before solving (4b) for Φ, it should be noted that if we put U (and ζ_I) equal to zero the terms in the braces become independent of (x,t), and thus we recover a delta function for Φ which integrates to produce the familiar gravity-wave dispersion formula. For the case $U \neq 0$, we shall approximate Φ by an exponential function and evaluate the integral equation by asymptotic means ($\mu \to \infty$).

3. Ordinary asymptotic solution

In order to illustrate the asymptotic procedure, we shall first use arguments based on ordinary stationary-phase methods. We shall obtain a solution for ζ_s by a method equivalent to solving the energy and wavenumber conservation equations (Phillips 1966, §§3.5–3.7). In the next section we shall examine the stationary-phase process in detail and extend it to cover the triple-root case.

The variation in y may be transformed out immediately. Noting that the argument of U_0 is $O(1)$, we find that k_1x and σt are $O(\mu)$ and $\zeta_I k$ is $O(1)$. Let

$$\Phi = a(k_1,k_2,\sigma)\exp[i\mu f(k_1,\sigma)]+O(\mu^{-1}), \quad (5)$$

where f and a are $O(1)$. Then, using standard stationary-phase techniques, i.e. $\partial^2 f/\partial k_1^2 \neq 0$ and $\partial^2 f/\partial\sigma^2 \neq 0$, (4$b$) can be integrated. If we force the two largest terms in the expansion in powers of μ^{-1} to be zero, we obtain an expression which relates the first partial derivatives† of f and also a partial differential equation for a. These reduce to ordinary derivatives for the special case $U_0 = U(x-ct)$, with c constant, and we shall now restrict our attention to this case only. It is apparent that, unless worse singularities arise when c is variable, the form of the solution for our special case will be all that is needed to represent the general solution in a uniformly valid sense. (The form $U(x-ct)$ is, after all, a valid subset of $U(x,t)$. It is not expected that worse singularities will occur, although we have not solved the problem for general $U(x,t)$.)

Let us redefine the co-ordinates in terms of a moving axis; in this way we shall be able to transform out the time dependence directly from (4). Let

$$x-ct=\eta, \quad k_1c+\sigma=\omega,$$

$$\frac{\partial U}{\partial x}=\frac{dU}{d\eta}=-\frac{1}{c}\frac{\partial U}{\partial t}.$$

† With the notation $U_0 = U(x,t)$, the expression referred to is

$$gk = \{\sigma+k_1U(-\partial f/\partial k_1, -\partial f/\partial\sigma)\}^2.$$

Thus (5) becomes

$$\Phi = a(k_1, k_2, \omega)\exp[i\mu f(k_1)] + O(\mu^{-1})$$

and the stationary-phase solution to (4*b*) becomes

$$gk = \{\omega + k_1 U(-\mu f') - k_1 c\}^2, \tag{6a}$$

$$a(k_1, k_2, \omega) = P(k_2, \omega)\left\{\frac{f''(k_1)}{gk_1(k_1^2+k_2^2)^{-\frac{1}{2}} - 2\{\omega + k_1(U-c)\}\{U-c\}}\right\}^{\frac{1}{2}} \exp(-k\zeta_I), \tag{6b}$$

where the argument of U is $-\mu f'(k_1)$ and $P(k_2, \omega)$ is an unspecified 'constant' of integration.

Using these expressions in (4*a*), and retaining only the largest term in the expansion, gives

$$\zeta_s = -\frac{i}{g}\iint_{-\infty}^{+\infty} P(\omega, k_2)\exp\{ik_2 y + i\omega t\}\,dk_2\,d\omega \int_{-\infty}^{+\infty}[\omega + k_1(U-c)]$$
$$\times\left\{\frac{f''(k_1)}{gk_1/k - 2\{\omega + k_1(U-c)\}\{U-c\}}\right\}^{\frac{1}{2}} \exp\{ik_1 x + i\mu f(k_1)\}\,dk_1. \tag{7a}$$

The integration over k_1 provides a stationary-phase point at $x + \mu f'(k_1) = 0$, and if $f''(k_1) \neq 0$ for any x of interest,

$$\zeta_s = \iint_{-\infty}^{+\infty} P(\omega, k_2)\exp(i\omega t + ik_2 y)\,d\omega\,dk_2\left[\sum_{k_1\ \text{roots}}\left(\frac{\pi}{\mu}\right)^{\frac{1}{2}}\right.$$
$$\left.\times\frac{\exp[i\int k_1(x)\,dx \pm \frac{1}{4}i\pi]\,(k_1^2+k_2^2)^{\frac{1}{8}}}{\{k_1 g^{\frac{1}{2}}/2(k_1^2+k_2^2)^{\frac{3}{4}} - \pm(U-c)\}^{\frac{1}{2}}} + O(\mu^{-\frac{3}{2}})\right]. \tag{7b}$$

The choice of signs is determined by the sign of f'', and P must be such that ζ_s is real for any real case. (The phase term $\int k_1(x)\,dx$ is obtained by expressing $\mu f(k_1)$ as $\int \mu f'(k_1)\,dk_1$, which from the definition of the stationary phase is $-\int x\,dk_1$. This integration is done by parts and provides $-xk_1 + \int k_1(x)\,dx$.) From the definition of $f(k_1)$ we find that the k_1 roots are given by

$$g(k_1^2+k_2^2)^{\frac{1}{2}} = (\omega + k_1[U(\eta) - c])^2 \tag{7c}$$

and this provides four real roots, one of which is always distinct and three of which may coalesce (Gargett & Hughes 1972). If there is a coalescence for some η (and k_2, ω, c), $f''(k_1) = 0$ at that point and the stationary-phase approximations break down. It should be noted that (7*b*) precisely defines the action-conserving solution of Longuet-Higgins & Stewart (1961) and Bretherton & Garrett (1968): the eighth root in the numerator of the integrand is the square root of the local wave frequency, and the denominator is the square root of the energy transmission velocity. Therefore the average square of the transform (in k_2, ω) of ζ_s, which is the energy spectrum multiplied by the energy velocity and divided by the local frequency, is a constant.

We shall now outline the solution steps necessary when $f'' = 0$ somewhere. In fact we shall analyse the case where $f''' = 0$ at the same location: this represents the worst singularity† and its uniformly valid solution contains the case $f''' \neq 0$.

4. Triple-root asymptotic method

The usual stationary-phase approximation procedure is to change the variable of integration, in this case from k_1 to u, such that the phase of the rapidly oscillating exponential contains powers of u up to u^2, and such that the point of stationary phase is at $u = 0$. Thus, in (7 a) we should have

$$k_1 x + \mu f(k_1) = \mu(\tfrac{1}{2}u_0^2 + \tfrac{1}{2}u^2)$$

and

$$x + \mu f'(k_1) = 0 \quad \text{at} \quad u = 0.$$

With this transformation the rest of the integrand can be expanded in a power series in u which can be integrated term by term. It becomes apparent that the nth term is $O(\mu^{-\frac{1}{2}(n+1)})$ and therefore this series is the desired asymptotic expansion. In the application of this method to the determination of Φ, the steps in the process can be clarified as follows. Let k_0 be a representative surface wavenumber scale for k and let l be a dimensionless distance $O(1)$, then

$$l = k_I x, \quad q = k_1/k_0, \quad \mu = k_0/k_I, \quad \psi(q, l) = ql + f(q).$$

Also let the integrand in (4) be

$$F_1(l, q) + \mu^{-1} F_2(l, q).$$

Then a and f are defined from the solution of

$$\int_{-\infty}^{+\infty} a(q) \exp\left[i\mu\psi(q, l)\right] \left(F_1 + \frac{F_2}{\mu}\right) dq = O\left(\frac{1}{\mu^2}\right). \tag{8a}$$

As indicated above, to establish the method let

$$\psi(q, l) = \tfrac{1}{2}u_0^2 + \tfrac{1}{2}u^2, \tag{9}$$

then

$$(\partial\psi/\partial q)\, dq = u\, du$$

and

$$\int_{-\infty}^{+\infty} \exp\left(i\mu \frac{u^2}{2}\right) \left(\frac{aF_1 + aF_2/\mu}{\partial\psi/\partial q}\right)_{q(u)} u\, du \sim 0. \tag{8b}$$

Let

$$\left(\frac{aF_1 + aF_2/\mu}{\partial\psi/\partial q}\right)_{q(u)} = \sum_{n=0}^{\infty} d_n u^n.$$

Then the integration can be performed and we find

$$d_0 + id_2/\mu + \ldots = 0. \tag{10}$$

† If surface tension is included, it can be shown that four roots may coalesce. With $g = 981\ \text{cm/s}^2$ and surface tension $= 70\ \text{cm}^2/\text{s}^2$, the parameters at coalescence are $k_1 = 0{\cdot}945$ rad/cm, $k_2 = \pm 0{\cdot}421$ rad/cm, $\omega = -17{\cdot}301$ rad/s and $U - c = -16{\cdot}66$ cm/s and the surface-wave propagation direction is at $\pm 24°$ with respect to the internal-wave direction. For a triple root in the absence of surface tension, only the propagation direction is fixed absolutely (at $\pm \tan^{-1}(0{\cdot}5)^{\frac{1}{2}} \approx \pm 35{\cdot}3°$).

Also, from the expansion,

$$d_0 = \lim_{u \to 0} \left\{ u \frac{aF_1 + aF_2/\mu}{\partial\psi/\partial q} \bigg|_{q(u)} \right\},$$

$$2d_2 = \lim_{u \to 0} \frac{d^2}{du^2} \left\{ u \frac{aF_1 + aF_2/\mu}{\partial\psi/\partial q} \bigg|_{q(u)} \right\}, \quad \text{etc.}$$

From the definition of ψ we find

$$\lim_{u \to 0} u(\partial\psi/\partial q)^{-1} \to |f''(q_0)|^{-\frac{1}{2}},$$

where

$$f'(q_0) + l = 0. \tag{11a}$$

By equating to zero the two largest powers of μ^{-1} in (10) we obtain the solution for the usual stationary-phase approximation:

$$F_1(q_0) = 0, \tag{11b}$$

$$a(q_0) = P \frac{f''(q_0)}{(\partial F_1/\partial q)_{l=l(q_0)}} \exp\left(-\tfrac{1}{2}k\zeta_I\right). \tag{11c}$$

The assumption that $f'' \neq 0$ is implicit in the form of (9), and we must now modify this equation suitably to account for the correct behaviour of f. Since we know that $f''' = 0$ somewhere, we must include powers of u of up to fourth order:

$$\psi(q, l) = s_0 + \left(\tfrac{1}{4}u^4 - \tfrac{1}{2}s_1 u^2 - s_2 u\right),$$

$$(f'(q, l) + l)\, dq = (\partial\psi/\partial q)\, dq = (u^3 - s_1 u - s_2)\, du,$$

where $s_{0,1,2}$ are parameters to be determined later and the three coalescing roots of $f'(q, l) + l = 0$ are chosen to coincide with the three roots of $u^3 - s_1 u - s_2 = 0$. (For a further extension of this concept see Bleistein (1967) and Ludwig (1966).) The integral in (8b) becomes

$$\int_{-\infty}^{+\infty} \exp\left[i\mu\left(\tfrac{1}{4}u^4 - \tfrac{1}{2}s_1 u^2 - s_2 u\right)\right] \left(\frac{\{aF_1 + aF_2/\mu\}}{\partial\psi/\partial q}\right)_{q=q(u)} (u^3 - s_1 u - s_2)\, du = O\left(\frac{1}{\mu^2}\right).$$

To evaluate this we must again expand the non-exponential part of the integrand, and the expansion which results in an asymptotic series in increasing powers of μ^{-1} is (Ursell 1972)

$$\left(\frac{aF_1 + aF_2/\mu}{\partial\psi/\partial q}\right)_{q(u)} (u^3 - s_1 u - s_2) = \sum_{m=0}^{\infty} (u^3 - s_1 u - s_2)^m (\alpha_m + u\beta_m + u^2\gamma_m).$$

As in the single-root case, we may equate the leading terms to zero after integration. We then find after much algebra that the equations for f', F_1 and a are exactly as before, i.e. (11 a–c), with the evaluation point q_0 being one of the three solutions to (11 a).

Using these values for f and a, and also using this triple-root expansion method, we may perform the k_1 integration for ζ_s. With the same scaling and with other non-dimensional parameters defined by

$$\Omega = \omega/(gk_0)^{\frac{1}{2}}, \quad r = k_2/k_0, \quad \xi = (gk_0)^{\frac{1}{2}}, \quad V_* = (U - c)(k_0/g)^{\frac{1}{2}}, \quad W_* = W(k_0/g)^{\frac{1}{2}},$$

$$m = k_I y, \quad \zeta_* = \zeta_s k_0, \quad \zeta_{*I} = k_0 \zeta_I,$$

we obtain

$$\zeta_* = -i\iiint_{-\infty}^{-\infty} \exp\left[i\Omega\xi + i\mu rm + i\mu ql + i\mu f(q) + \tfrac{1}{2}\zeta_{*I}(q^2+r^2)^{\frac{1}{2}}\right] \times \left[p + qV_* - iW_*(q^2+r^2)^{\frac{1}{2}}\right]\left(\left(\frac{f''(q)}{\partial F_1/\partial q}\right)^{\frac{1}{2}} + O(\mu^{-1})\right) dq\,dr\,d\Omega. \quad (12)$$

As before, $$f(q) + ql \equiv \psi(q) = s_0 + (\tfrac{1}{4}u^4 - \tfrac{1}{2}s_1 u^2 - s_2 u)$$

and we expand the appropriate part of the integrand in the proper power series†

$$\left[p + qV_* - iW_*(q^2+r^2)^{\frac{1}{2}}\left(\frac{f''(q)}{\partial F_1/\partial q}\right)^{\frac{1}{2}}\left[\frac{u^3 - s_1 u - s_2}{\partial\psi/\partial q}\right]_{Q(u)} \exp\left[\tfrac{1}{2}\zeta_{*I}\{q^2+r^2\}^{\frac{1}{2}}\right]\right.$$
$$= \sum_{m=0}^{\infty} (\alpha_m + \beta_m u + \gamma_m u^2)(u^3 - s_1 u - s_2)^m.$$

On integration, this yields

$$\zeta_* = -i\iint_{-\infty}^{+\infty} P(\Omega, r)\exp(i\Omega\xi + i\mu rm + i\mu s_0) \left[\frac{\alpha_0 I_0^*}{\mu^{\frac{1}{4}}} + \frac{\beta_0 I_1^*}{\mu^{\frac{1}{2}}} + \frac{\gamma_0 I_2^*}{\mu^{\frac{3}{4}}} + O\left(\frac{1}{\mu}\right)\right] d\Omega\,dr,$$

with the three integrals I_n^* defined as follows (see also appendix):

$$I_n^* = \int_{-\infty}^{+\infty} v^n \exp\left[i(\tfrac{1}{4}v^4 - \tfrac{1}{2}\mu^{\frac{1}{2}} s_1 v^2 - \mu^{\frac{3}{4}} s_2 v)\right] dv, \quad n = 0, 1, 2. \quad (13)$$

The values of α_0, β_0 and γ_0 are obtained from the simultaneous solution of

$$\alpha_0 + u_1\beta_0 + u_1^2\gamma_0 = \frac{(q_1^2+r^2)^{\frac{1}{8}}}{2^{\frac{1}{2}}[q_1/2(q_1^2+r^2)^{\frac{3}{4}} - V_*]^{\frac{1}{2}}}\{3u_1^2 - s_1\}^{\frac{1}{2}}, \quad (14a)$$

$$\alpha_0 + u_2\beta_0 + u_2^2\gamma_0 = \frac{(q_2^2+r^2)^{\frac{1}{8}}}{2^{\frac{1}{2}}[q_2/2(q_2^2+r^2)^{\frac{3}{4}} - V_*]^{\frac{1}{2}}}\{3u_2^2 - s_1\}^{\frac{1}{2}}, \quad (14b)$$

$$\alpha_0 + u_3\beta_0 + u_3^2\gamma_0 = \frac{(q_3^2+r^2)^{\frac{1}{8}}}{2^{\frac{1}{2}}[q_3/2(q_3^2+r^2)^{\frac{3}{4}} - V_*]^{\frac{1}{2}}}\{3u_3^2 - s_1\}^{\frac{1}{2}}, \quad (14c)$$

with $u_{1,2,3}^3 - s_1 u_{1,2,3} - s_2 = 0$ defining the three u roots. The three q roots are defined by

$$(r^2 + q_n^2)^{\frac{1}{2}} = \{(p + q_n V_*(l))\}^2. \quad (15)$$

Finally, the three s values are given by the three equations

$$f(q_n) + lq_n = s_0 + \tfrac{1}{4}u_n^4 + \tfrac{1}{2}s_1 u_n^2 + s_2 u_n. \quad (16)$$

† It is possible to generalize this equation by rewriting each side as an infinite power series (in μ^{-1}) with 'known' coefficients on the left-hand side and α_m, β_m and γ_m each becoming an infinite power series (in μ^{-1}). For the present problem we shall determine only the largest terms (three in this case), and will omit terms $O(\mu^{-1})$. In the interest of simplicity, we shall ignore these terms first and thus circumvent the necessity for making the general expansion. To determine the coefficients on the right-hand side, we repeatedly differentiate with respect to u and evaluate both sides at the positions of the three roots. (The same process applied to a double root is examined in detail by Chester, Friedman & Ursell 1957.)

Since $f'(q_n) = -l$ defines q_n as a function of l, integration by parts produces

$$f(q_n) + lq_n = \int q_n(l)\,dl. \tag{17}$$

This completes the solution. Using (15), (17) and (16) and the defining cubic for $u_{1,2,3}$, all parameters in (14) can be specified, and α_0, β_0 and γ_0 can be determined as functions of l (and r, p).

Further algebraic manipulations of the three equations in (16) can be performed using the properties of the u_n. From the cubic we have $u_1+u_2+u_3 = 0$ and $u_1 u_2 u_3 = s_2$, and, using similar relations, it is possible to determine s_0, s_1 and s_2 separately in terms of the right-hand sides of (16). The expressions are unwieldy and will not be reproduced here.

The coefficients α_0, β_0 and γ_0 are almost everywhere finite, since the zeros in the denominators of the right-hand sides of (14) occur at the same locations as the zeros in the numerators. (The coefficients are not finite if a double or triple root coincides with a zero in the gradient of V_*; this case is discussed in the next section.)

Figure 1 illustrates of the behaviour of q as a function of l (i.e. k_1/k_0 as a function of $x-ct$) for a velocity field which is sinusoidal (0° to 90° shown), and with the other parameters chosen to give (*a*) a single root everywhere, (*b*) two separated double roots and (*c*) a triple root. The velocity U is also given.

The three curves are solutions of the dispersion relation (16) [whose dimensional form is (7 *c*)] for the parameter values given in the caption. The ordinary stationary-phase approximations are valid if significant variations in wavenumber and amplitude take place over distances which are large compared with the wavelength, thus, in particular, they break down where $dk_1/d(x-ct) = \infty$. For curve (*a*) this does not occur and ordinary stationary-phase approximations are valid everywhere. For curve (*b*) this occurs twice (at the double roots) and for curve (*c*) once (at the triple root), and we need the extended stationary-phase solution to provide meaningful estimates for ζ_s in these regions. Figure 2 depicts the amplitude of the component for each of the three cases in figure 1; the amplitude ζ is defined as $|\alpha_0 I_0^* \mu^{-\frac{1}{3}} + \beta_0 I_1^* \mu^{-\frac{2}{3}} + \gamma_0 I_2^* \mu^{-\frac{4}{3}}|$ and is normalized for each case by the amplitude at $U = 0$.

In general, the amplitude at the triple root is dominated by the first term, and it can be shown that for an arrangement of constants such that

$$\left|\frac{\alpha_0 I_0^*}{\mu^{\frac{1}{3}}} + \frac{\beta_0 I_1^*}{\mu^{\frac{2}{3}}} + \frac{\gamma_0 I_2^*}{\mu^{\frac{4}{3}}}\right| \to \frac{\sigma^{\frac{1}{2}}}{[C_{g1} + U - c]^{\frac{1}{2}}}$$

far from a multiple root, i.e. the constant-action form, at the triple root this form becomes

$$\left|\frac{\alpha_0 I_0^*}{\mu^{\frac{1}{3}}}\right| \to \frac{1{\cdot}0682\sigma^{\frac{1}{2}}}{[C_{g1}\, U'(\eta)/k_1]^{\frac{1}{3}}}. \tag{18}$$

5. Asymptotic order of magnitude for a multiple root near $U' = 0$

It can be seen from (18) that if $U' = 0$ at a multiple root our solution still possesses a singularity. The reason for this breakdown is the implicit assumption that U' exists when defining f' from the inverse function of U. If U' is zero at the

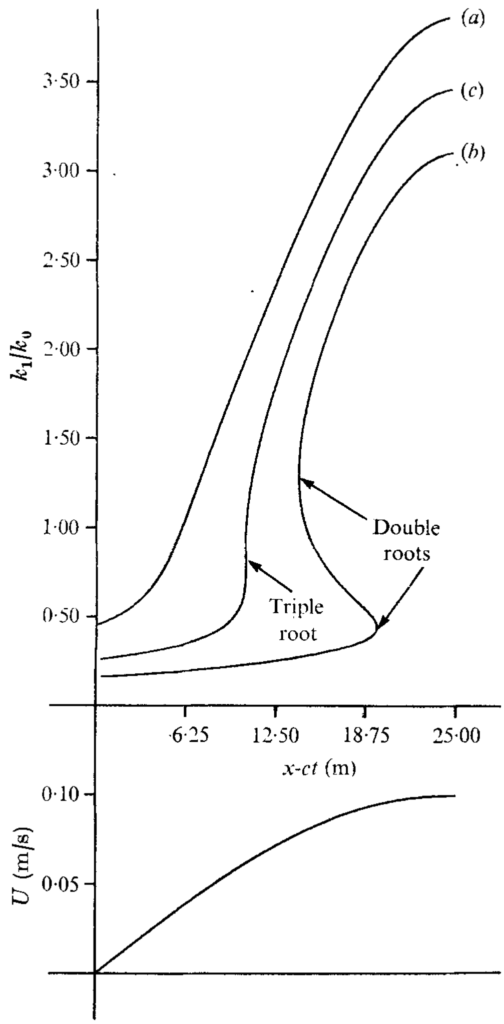


FIGURE 1. Variation of wavenumber for a particular current. Curve (a) illustrates a single-root case, (b) two double roots and (c) a triple root. $c = 0{\cdot}33$ m/s, $k_I = 64$ rad/km, $k_0 = 22{\cdot}52$ rad/m, $k_2 = -13{\cdot}016$ rad/m, $\mu = 351{\cdot}9$. For (a) $\omega = -9{\cdot}364$ rad/s, for (b) $\omega = -10{\cdot}33$ rad/s and for (c) $\omega = -9{\cdot}914$ rad/s.

root position, a quartic is not sufficient to define f uniformly. The problem may also be seen by referring to figure 1. Because of (6a) we may relabel the $x-ct$ axis as $-\mu f'(k)$. Thus the depicted curves can be considered as defining $f'(k)$ as a function k, although we then have f' plotted along the abscissa and k plotted along the ordinate. With this interpretation in mind we see that the three curves can all be described by a cubic in k and indeed this is one interpretation of why a quartic for f is needed for this problem. However, if the double root on (b) or the triple root on (c) coincided with the peak in U, i.e. $U' = 0$, the curvature of the k-curve at that point would be radically altered (even to the point of being cusplike, depending on the flatness of U) and terms with fractional powers

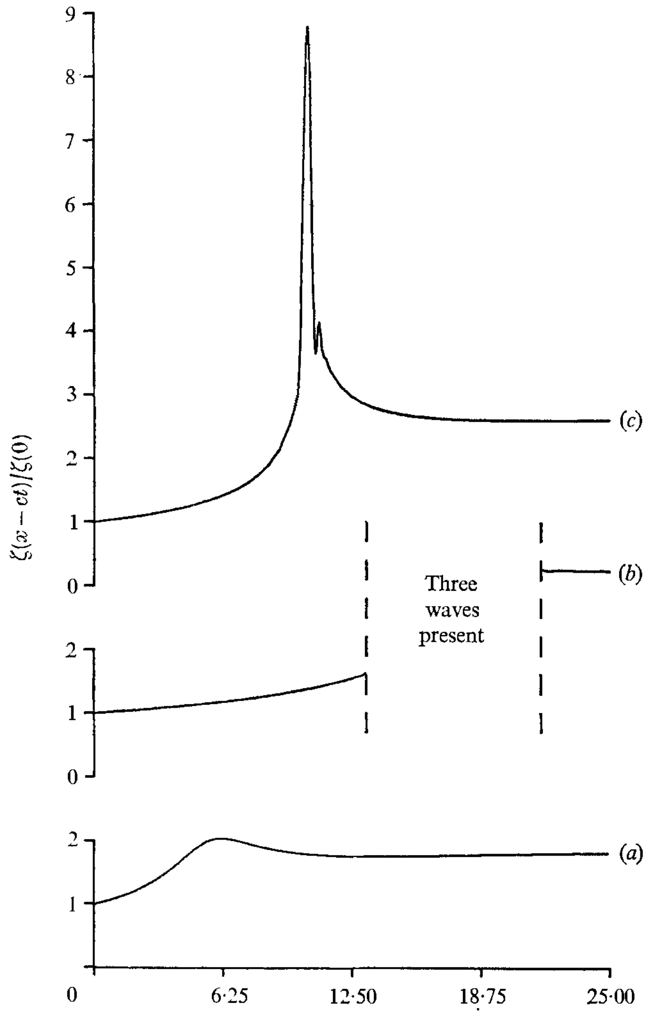


FIGURE 2. Amplitude dependence for the three cases shown in figure 1.

would be needed to specify the behaviour of f locally. Therefore, an adequate, uniformly valid specification of f requires a generic form which naturally incorporates all the 'loops', cusps and multiple-valuedness that are possible from solving (6 a).

We have not identified that generic form, and in fact we expect that it will have an infinite number of disposable constants allowing for the vanishing of all the derivatives of U up to any order at the multiple-root position. Rather than pursue this line any further, we shall show that the integral in (12) over q produces finite values for any regular behaviour of U at the triple point, and we shall give the order in μ of the three largest terms in the expansion. With the non-dimensionalized form of the variables used in (12) let $V_*(l) = V_m - l^n$. Then the location of interest

Term \ n	0	1	2	3	∞
First	$-1/2$	$-1/4$	$-1/10$	0	$+1/2$
Second	$-1/2$	$-1/2$	$-1/2$	$-1/2$	$-1/2$
Third	$-1/2$	$-3/4$	$-9/10$	-1	$-3/2$

TABLE 1. Exponents of μ in expression for I. The first term is for the coefficient of γ_1, the second for the coefficient of γ_2 and the third for the coefficient of γ_3. As the velocity behaviour becomes more strongly varying at the triple-root location, i.e. as $n \to \infty$, the second and third term become progressively less important than the first and the first term itself becomes progressively larger.

is $l = 0$, and it is known that, for the triple root $q^2 = 2r^2$, $\Omega = -(16r^2/27)^{\frac{1}{4}}$ (Gargett & Hughes 1972). Thus

$$V_m = -(\{3r^2\}^{\frac{1}{2}} - \Omega)/2^{\frac{1}{2}}|r|,$$

and the integral in question is

$$I = -\frac{i}{2^{\frac{1}{2}}}\int_{-\infty}^{2^{\frac{1}{2}}|r|} \frac{\exp\left\{i\mu\left[lq - \int^{q}\left(V_m - \dfrac{\pm\{q^2+r^2\}^{\frac{1}{2}} - \Omega}{q}\right)\right]^{1/n} dq\right\}}{\{q^4(q^2+r^2)\}^{\frac{1}{4}}}$$
$$\times (p + qV_*)\left[\frac{\pm 1}{n\left(V_m - \dfrac{\pm\{q^2+r^2\}^{\frac{1}{2}} - \Omega}{q}\right)^{1-(1/n)}}\right]^{\frac{1}{2}} dq.$$

Let $q = 2^{\frac{1}{2}}|r| + \epsilon$. Then a straightforward expansion of the terms in the integrand in powers of ϵ produces the result that the largest terms behave as

$$I = B\int_{-\infty}^{0} \epsilon^{3/2n-\frac{3}{2}} \exp\{i\mu(\epsilon l - A\epsilon^{3/n+1})\}\, d\epsilon,$$

where A and B are functions of Ω and r. By a direct application of the method of Bleistein (1967), we find

$$I \sim \gamma_1\mu^{-(3-n)/(6+2n)} + \gamma_2\mu^{-\frac{1}{2}} + \gamma_3\mu^{-(3+3n)/(6+2n)} + \dots.$$

Table 1 summarizes the behaviour of the exponents of μ as functions of n.

6. Summary

One traditional analytic approach to the problem of nonlinearly interacting waves is to examine the possible resonances that can occur between vanishingly weak waves. In our case we should find a *difference* interaction occurring between a triad of waves two of which are surface waves with almost equal wavenumbers, the third being an internal wave with a much smaller wavenumber. The variation in surface-wave amplitude would then be ascribed to beats between the two surface waves and as such would necessarily be sinusoidal for a sinusoidal internal-wave field. We have solved this particular form of interaction for the relaxed condition that the internal wave is not vanishingly weak.

A second traditional analytical approach to this particular problem (disparate wavenumbers and frequencies, quadratic interaction) is to use the dynamical conservation equation and interpret the small wavenumber, small frequency wave as perturbing the medium in which the other wave(s) are propagating. This approach is equivalent to the WKB approximation and for very weak perturbations leads to the same results as the resonance theory. It is known to break down at caustics but it does have the advantage of being able to treat non-vanishingly weak perturbations. In our method of solution we have extended the WKB technique to cover the 'worst' caustic situation present in this interaction problem (for $\partial U/\partial x \neq 0$), namely two caustics which have merged into a cusp, and we have given the solution in terms of Pearcey functions.

We find that we require the simultaneous existence of three modulated surface waves with coupled amplitudes and phases. The ordinary WKB solution allows these to be uncoupled and thus independently specified (in terms of boundary or initial values). As an example (using heuristic arguments), in figures 1 and 2, curve (*b*), we have energy flowing along the curve in a zig-zag fashion from left to right; thus, since the flux of action must be constant, this insists on the proper coupling of the amplitudes of the three waves in the region between the double roots. For the triple root (curve *c*) only one real root exists everywhere, however we again expect the action flux to be constant along the curve. This specifies the amplitude to the right of the singularity in terms of its value to the left. Of course, 'action' and 'energy' fluxes are poorly defined in terms of sinusoids right at the singularities, but our extensions to the theory using Pearcey functions precisely circumvent this difficulty.

Finally, at the triple-root point we find that $\zeta_s^4\,\partial U/\partial x$ is bounded if $\partial U/\partial x$ is nonzero, and $\zeta_s(\partial^n U/\partial x^n)^{(3-n)/(6+2n)}$ is bounded if $\partial U/\partial x = \partial^2 U/\partial x^2 = \partial^{n-1} U/\partial x^{n-1} = 0$.

I am indebted to R. Smith (Cambridge University) for bringing to my attention the original publication by Pearcey (1946) and for an interesting and fruitful correspondence on this whole subject. The program for calculating the three Pearcey functions was written and tested by R. Csomany. The examples that are illustrated were programmed and the graphs were prepared by R. S. Anderson. It is with great pleasure that I record my appreciation to them. The material covered in this paper has appeared as an appendix in DREP Report no. 75-3.

Appendix. Behaviour of the three basic integrals (Pearcey functions)

If we define the three integrals as

$$I_n(x,y) = \int_{-\infty}^{+\infty} v^n \exp\left[i(\tfrac{1}{4}v^4 - \tfrac{1}{2}xv^2 - yv)\right] dv,$$

then $I_1(x,y) = i\,\partial I_0/\partial y$, $I_2(x,y) = 2i\,\partial I_0/\partial x$ and, in fact, $\partial I_0/\partial x = -\tfrac{1}{2}i\partial^2 I_0/\partial y^2$ with a Bessel-function boundary condition on $y = 0$. (See also Pearcey (1946), who has previously discussed the case $n = 0$ and who provides values of $I_0(-x, -2^{\frac{1}{2}}y)/2^{\frac{1}{2}}$.)

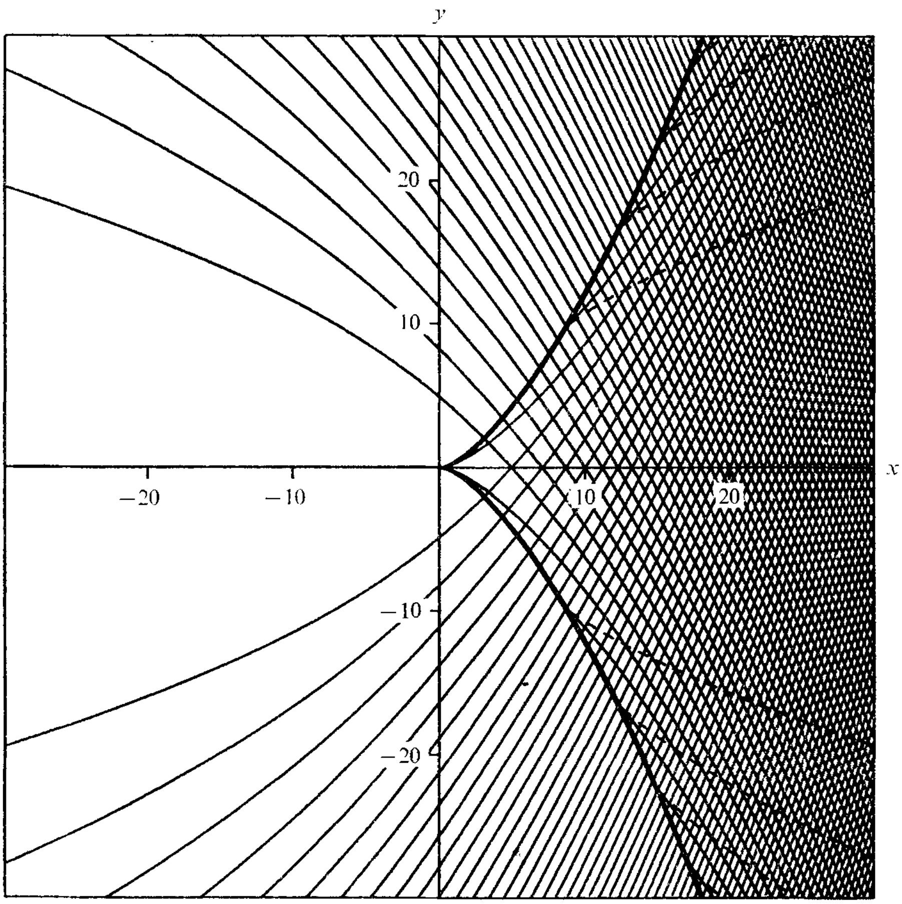


FIGURE 3. Isophase lines for $I_n(x, y)$ calculated by ordinary method of stationary phase. The separation in phase between lines of the same set is 2π.

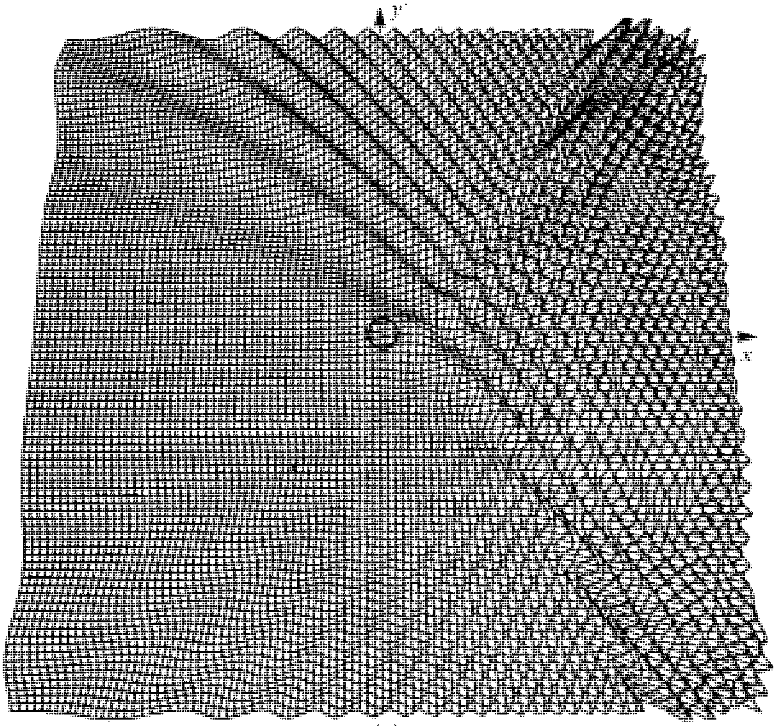
Using the ordinary stationary-phase method, we find one wave system if $y^2 > \frac{4}{27}x^3$, three wave systems for $y^2 < \frac{4}{27}x^3$ and a coalescence for $y^2 = \frac{4}{27}x^3$. If we exclude the region near the coalescence line (and the origin), we can approximate the integrals by

$$\frac{I_n(x,y)}{\pi^{\frac{1}{2}}} \approx \sum_{j=1}^{3} v_j^n \frac{\exp\left[i(\tfrac{1}{4}v_j^4 - \tfrac{1}{2}xv_j^2 - yx_j) + \tfrac{1}{4}i\pi\right]}{(3v_j^2 - x)^{\frac{1}{2}}}, \quad n = 0, 1, 2,$$

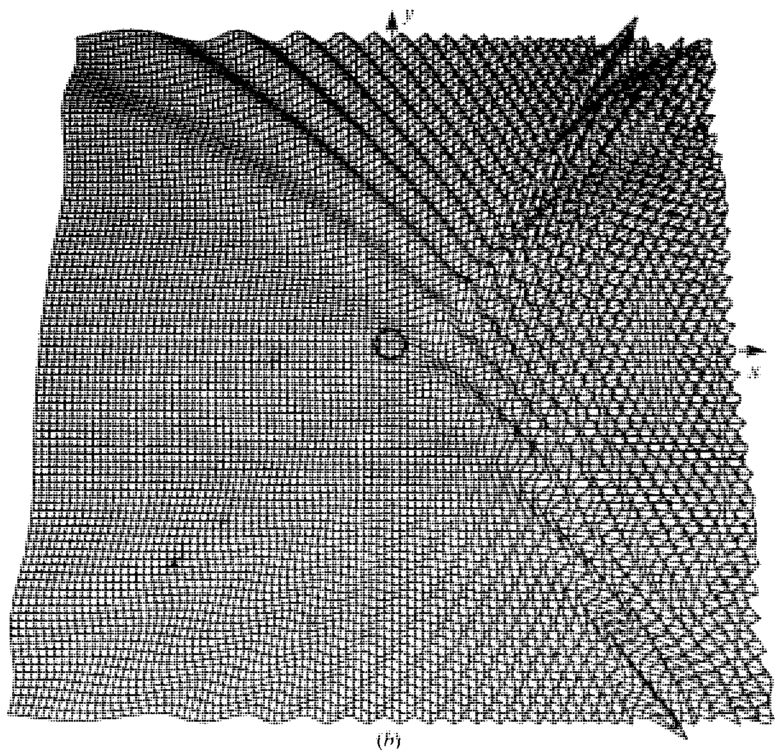
$$v_j^3 - xv_j - y = 0, \quad j = 1, 2, 3.$$

Using these expressions, we have determined isophase lines for each of the three terms; these are depicted in figure 3. One term is represented by the lines drawn from the upper left to lower right, another by the symmetric set from the lower left to upper right, and the third term by the dashed lines. The heavy line is the coalescence curve. Along the y axis (one real root, $v = y^{\frac{1}{3}}$),

$$|I_0(0,y)| \sim (\tfrac{1}{3}\pi)^{\frac{1}{2}}\,|y|^{-\frac{1}{3}}, \quad |I_1(0,y)| \sim (\tfrac{1}{3}\pi)^{\frac{1}{2}}, \quad |I_2(0,y)| \sim (\tfrac{1}{3}\pi)^{\frac{1}{2}}\,|y|^{\frac{1}{3}}.$$



(a)



(b)

FIGURE 4. Perspective view of (a) real part and (b) imaginary part of $I_0(x, y)$. The centre of the circle is the origin, and the range of x and y is -20 to $+20$.

Along any line of constant phase, the relevant term produces the following behaviour for large x:

$$
\begin{aligned}
|I_0(x,y)| &\sim \pi^{\frac{1}{2}}\,|x|^{-\frac{1}{2}},\\
|I_1(x,y)| &\sim \begin{cases} (\frac{2}{3}\pi)^{\frac{1}{2}} & \text{for} \quad x > 0,\\ (2\pi\mathrm{ph})^{\frac{1}{2}}\,|x|^{-1} & \text{for} \quad x < 0,\end{cases}\\
|I_2(x,y)| &\sim \begin{cases} \frac{2}{3}\pi^{\frac{1}{2}}\,|x|^{\frac{1}{2}} & \text{for} \quad x > 0,\\ 2\mathrm{ph}\pi^{\frac{1}{2}}\,|x|^{-\frac{3}{2}} & \text{for} \quad x < 0,\end{cases}
\end{aligned}
$$

where ph, the phase, is the constant value of $\frac{1}{4}v_j^4 - \frac{1}{4}xv_j^2 - \frac{1}{2}xv_j^2 - yv_j$. Using Airy functions, the description of the I_n may be extended to the situation where either x or y is large. In particular, along the coalescence line, the double-root terms behave as

$$
\begin{aligned}
|I_0(3(\tfrac{1}{2}|y|)^{\frac{3}{2}}, y)| &\sim 2\pi\mathrm{Ai}(0)\,2^{\frac{1}{4}}3^{-\frac{1}{3}}\,|y|^{-\frac{1}{4}},\\
|I_1(3(\tfrac{1}{2}|y|)^{\frac{3}{2}}, y)| &\sim 2\pi\mathrm{Ai}(0)\,2^{-\frac{1}{12}}3^{-\frac{1}{3}}\,|y|^{\frac{1}{12}},\\
|I_2(3(\tfrac{1}{2}|y|)^{\frac{3}{2}}, y)| &\sim 2\pi\mathrm{Ai}(0)\,2^{-\frac{5}{12}}3^{-\frac{1}{3}}\,|y|^{\frac{5}{12}}.
\end{aligned}
$$

The real and imaginary parts of $I_0(x,y)$ are shown in perspective in figures 4 (*a*) and (*b*).

Finally, at the origin we have

$$
\begin{aligned}
I_0(0,0) &= \frac{\pi}{2^{\frac{3}{2}}\,\Gamma(\frac{3}{4})}\left[\frac{1}{\cos\frac{3}{8}\pi} + \frac{i}{\sin\frac{3}{8}\pi}\right] \approx 2{\cdot}37 + 0{\cdot}981i,\\
I_1(0,0) &= 0,\\
I_2(0,0) &= \frac{\pi}{2\Gamma(\frac{1}{4})}\left[\frac{1}{\cos\frac{1}{8}\pi} + \frac{i}{\sin\frac{1}{8}\pi}\right] \approx 0{\cdot}663 + 1{\cdot}60i.
\end{aligned}
$$

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